1. **(15 points)** *For the following system of equations, determine its solution set or describe it as inconsistent:*

$$\begin{cases} x_1 - 3x_2 - 6x_3 - 5x_4 = -13 \\ 2x_1 + x_2 + 2x_3 + 11x_4 = 9 \\ 3x_1 + x_2 + 2x_3 + 15x_4 = 11 \end{cases}$$

Performing Gauss-Jordan elimination on the associated augmented matrix:

$$\left(\begin{array}{cccc|c} 1 & -3 & -6 & 5 & -13 \\ 2 & 1 & 2 & 11 & 9 \\ 3 & 1 & 2 & 15 & 11 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -3 & -6 & 5 & -13 \\ 0 & 7 & 14 & 1 & 35 \\ 0 & 10 & 20 & 0 & 50 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & -3 & -6 & 5 & -13 \\ 0 & 1 & 2 & \frac{1}{7} & 5 \\ 0 & 10 & 20 & 0 & 50 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & \frac{38}{7} & 2 \\ 0 & 1 & 2 & \frac{1}{7} & 5 \\ 0 & 0 & 0 & \frac{-10}{7} & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & \frac{38}{7} & 2 \\ 0 & 1 & 2 & \frac{1}{7} & 5 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 & 5 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

From the above we can conclude that $x_1 = 2$, $x_4 = 0$, and that $x_2 + 2x_3 = 5$. For the last of these three criteria, we may either put x_2 in terms of x_3 or vice versa; doing the former gives us the solution

$$(x_1, x_2, x_3, x_4) = (2, 5 - 2x_3, x_3, 0)$$

where x_3 could be any real number.

2. **(10 points)** *Let* $A = \begin{pmatrix} 2 & 0 & -4 \\ 3 & 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 \end{pmatrix}$, *and* $\mathbf{v} = \begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix}$. *For each of the following arithmetic expressions, either calculate its value or explain briefly why it cannot be calculated.*

(a) $B\mathbf{v}$.

Since B has 2 columns and $\mathbf{v}$ has 3 rows, this matrix multiplication is not valid.

(b) $A\mathbf{v}$.

$$A\mathbf{v} = \begin{pmatrix} 2 & 0 & -4 \\ 3 & 1 & 0 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \cdot 5 + 0 \cdot 2 - 4 \cdot 0 \\ 3 \cdot 5 + 1 \cdot 2 + 0 \cdot 0 \end{pmatrix} = \begin{pmatrix} 10 \\ 17 \end{pmatrix}$$

(c) $2A + (\mathbf{v}B)^T$.

This complicated expression is (perhaps surprisingly) valid: The product of the 3×1 matrix $\mathbf{v}$ and the 1×2 matrix B is a 3×2 matrix, with a 2×3 transpose which is compatible under addition with a multiple of A. More specifically:

$$2A+(\mathbf{v}B)^T = 2\begin{pmatrix}2 & 0 & -4\\ 3 & 1 & 0\end{pmatrix}+\left(\begin{pmatrix}5\\2\\0\end{pmatrix}\begin{pmatrix}1 & -1\end{pmatrix}\right)^T = \begin{pmatrix}4 & 0 & -8\\ 6 & 2 & 0\end{pmatrix}+\begin{pmatrix}5 & -5\\ 2 & -2\\ 0 & 0\end{pmatrix}^T = \begin{pmatrix}7 & 2 & -8\\ 1 & 0 & 0\end{pmatrix}$$

(d) BA.

$$BA = \begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}2 & 0 & -4\\ 3 & 1 & 0\end{pmatrix} = \begin{pmatrix}1\cdot 2-1\cdot 3 & 1\cdot 0-1\cdot 1 & 1\cdot(-4)-1\cdot 0\end{pmatrix} = \begin{pmatrix}-1 & -1 & -4\end{pmatrix}$$

3. **(20 points)** *Let* $A = \begin{pmatrix}2 & 1 & 2\\ 3 & 2 & 2\\ 1 & 2 & 3\end{pmatrix}$.

(a) **(12 points)** *Calculate the inverse of* A.

Plausible approaches to this calculation include converting the augmented matrix $\left(\begin{array}{c|c} A & I\end{array}\right)$ to $\left(\begin{array}{c|c} I & A^{-1}\end{array}\right)$ with elementary row operations, or using the adjoint formula for a matrix inverse. The former approach is shown first, using some nonstandard choices of elementary row operations to avoid unpleasant intermediary fractions:

$$\left(\begin{array}{ccc|ccc}2 & 1 & 2 & 1 & 0 & 0\\ 3 & 2 & 2 & 0 & 1 & 0\\ 1 & 2 & 3 & 0 & 0 & 1\end{array}\right) \sim \left(\begin{array}{ccc|ccc}1 & 2 & 3 & 0 & 0 & 1\\ 2 & 1 & 2 & 1 & 0 & 0\\ 3 & 2 & 2 & 0 & 1 & 0\end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc}1 & 2 & 3 & 0 & 0 & 1\\ 0 & -3 & -4 & 1 & 0 & -2\\ 0 & -4 & -7 & 0 & 1 & -3\end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc}1 & 2 & 3 & 0 & 0 & 1\\ 0 & 1 & 3 & 1 & -1 & 1\\ 0 & -4 & -7 & 0 & 1 & -3\end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc}1 & 0 & -3 & -2 & 2 & -1\\ 0 & 1 & 3 & 1 & -1 & 1\\ 0 & 0 & 5 & 4 & -3 & 1\end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc}1 & 0 & -3 & -2 & 2 & -1\\ 0 & 1 & 3 & 1 & -1 & 1\\ 0 & 0 & 1 & \frac{4}{5} & -\frac{3}{5} & \frac{1}{5}\end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc}1 & 0 & 0 & \frac{2}{5} & \frac{1}{5} & -\frac{2}{5}\\ 0 & 1 & 0 & -\frac{7}{5} & \frac{4}{5} & \frac{2}{5}\\ 0 & 0 & 1 & \frac{4}{5} & -\frac{3}{5} & \frac{1}{5}\end{array}\right)$$

or alternatively, using the adjoint formula:

$$A^{-1} = \frac{1}{\det A}\operatorname{adj} A = \frac{1}{\begin{vmatrix}2 & 1 & 2\\ 3 & 2 & 2\\ 1 & 2 & 3\end{vmatrix}}\begin{pmatrix}\begin{vmatrix}2 & 2\\ 2 & 3\end{vmatrix} & -\begin{vmatrix}1 & 2\\ 2 & 3\end{vmatrix} & \begin{vmatrix}1 & 2\\ 2 & 2\end{vmatrix}\\ -\begin{vmatrix}3 & 2\\ 1 & 3\end{vmatrix} & \begin{vmatrix}2 & 2\\ 1 & 3\end{vmatrix} & -\begin{vmatrix}2 & 2\\ 3 & 2\end{vmatrix}\\ \begin{vmatrix}3 & 2\\ 1 & 2\end{vmatrix} & -\begin{vmatrix}2 & 1\\ 1 & 2\end{vmatrix} & \begin{vmatrix}2 & 1\\ 3 & 2\end{vmatrix}\end{pmatrix} = \frac{1}{5}\begin{pmatrix}2 & 1 & -2\\ -7 & 4 & 2\\ 4 & -3 & 1\end{pmatrix}$$

(b) **(5 points)** *Find a solution* $\mathbf{y}$ *to the matrix equation*

$$\begin{pmatrix} 2 & 1 & 2 \\ 3 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix} \mathbf{y} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

This can of course be solved by row-reducing the augmented matrix, but armed with the inverse from above, it is much easier to observe that since $A\mathbf{y} = \left(\begin{smallmatrix} 3 \\ -1 \\ 2 \end{smallmatrix}\right)$, it is the case that $\mathbf{y} = A^{-1}\left(\begin{smallmatrix} 3 \\ -1 \\ 2 \end{smallmatrix}\right)$ and so:

$$\mathbf{y} = \begin{pmatrix} \frac{2}{5} & \frac{1}{5} & -\frac{2}{5} \\ -\frac{7}{5} & \frac{4}{5} & \frac{2}{5} \\ \frac{4}{5} & -\frac{3}{5} & \frac{1}{5} \end{pmatrix} \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0.2 \\ -4.2 \\ 3.4 \end{pmatrix}$$

(c) **(3 points)** *Calculate the dimension of the nullspace of* A*, and determine a basis for it.* Since A is singular, the only solution to $A\mathbf{x} = \mathbf{0}$ is when $\mathbf{x} = \mathbf{0}$, so A has the single-element nullspace $\{\mathbf{0}\}$, which is zero-dimensional and has a basis consisting of no vectors.

4. **(5 points)** *Find a value* k *such that the matrix* $\begin{pmatrix} 2 & 1 & 0 \\ 3 & -5 & k \\ -1 & 2 & 1 \end{pmatrix}$ *is singular.*

A matrix is singular if and only if its determinant is zero, so we seek a value k such that

$$\begin{vmatrix} 2 & 1 & 0 \\ 3 & -5 & k \\ -1 & 2 & 1 \end{vmatrix} = 0$$

which will be the case when

$$-10 - k + 0 - 4k - 3 = 0$$

whose solution is $k = \frac{-13}{5}$.

This problem could also be solved (with more difficulty) by row-reducing the matrix to get something akin to $\begin{pmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{2}{13}k \\ 0 & 0 & \frac{5}{13}k+1 \end{pmatrix}$, which fails to have a pivot in the last row when $\frac{5}{13}k+1 = 0$, which is true when $k = \frac{-13}{5}$.

5. **(25 points)** *Answer the following questions.*

(a) **(10 points)** *Calculate the determinant* $\begin{vmatrix} 1 & -3 & 2 & 5 \\ 1 & -3 & 2 & 4 \\ 0 & 1 & -3 & 2 \\ 2 & -1 & 0 & 3 \end{vmatrix}$.

Since Type III row operations don't change the determinant, we can make this question massively simpler by subtracting the second row from the first (or vice versa):

$$\begin{vmatrix} 1 & -3 & 2 & 5 \\ 1 & -3 & 2 & 4 \\ 0 & 1 & -3 & 2 \\ 2 & -1 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & -3 & 2 & 4 \\ 0 & 1 & -3 & 2 \\ 2 & -1 & 0 & 3 \end{vmatrix}$$

and then use a cofactor expansion along the first row:

$$\begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & -3 & 2 & 4 \\ 0 & 1 & -3 & 2 \\ 2 & -1 & 0 & 3 \end{vmatrix} = 0 - 0 + 0 - 1 \cdot \begin{vmatrix} 1 & -3 & 2 \\ 0 & 1 & -3 \\ 2 & -1 & 0 \end{vmatrix} = -(0 + 18 + 0 - 3 - 0 - 4) = -11.$$

Other approaches such as reducing the matrix to a triangular form and recording type I and II moves would also work.

(b) **(8 points)** Calculate the value of x (and only x) in the following equation:

$$\begin{pmatrix} 1 & -3 & 2 & 5 \\ 1 & -3 & 2 & 4 \\ 0 & 1 & -3 & 2 \\ 2 & -1 & 0 & 3 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

Cramer's rule tells us that x has a value given by the ratio of determinants

$$\frac{\begin{vmatrix} 1 & 2 & 2 & 5 \\ 1 & 2 & 2 & 4 \\ 0 & 0 & -3 & 2 \\ 2 & 1 & 0 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & -3 & 2 & 5 \\ 1 & -3 & 2 & 4 \\ 0 & 1 & -3 & 2 \\ 2 & -1 & 0 & 3 \end{vmatrix}}$$

The denominator is known to be -11 from the previous result. The numerator can be calculated much the same way as the determinant in the previous question was found:

$$\begin{vmatrix} 1 & 2 & 2 & 5 \\ 1 & 2 & 2 & 4 \\ 0 & 0 & -3 & 2 \\ 2 & 1 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & 2 & 2 & 4 \\ 0 & 0 & -3 & 2 \\ 2 & 1 & 0 & 3 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 2 \\ 0 & 0 & -3 \\ 2 & 1 & 0 \end{vmatrix} = -3\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 9$$

so $x = \frac{9}{-11}$. Incidentally, $w = \frac{1}{11}$, $y = \frac{-3}{11}$, and $z = 0$ (although those were not requested in the problem).

6. **(7 points)** *What is the entry in the second row and third column of* $\begin{pmatrix} 1 & -3 & 2 & 5 \\ 1 & -3 & 2 & 4 \\ 0 & 1 & -3 & 2 \\ 2 & -1 & 0 & 3 \end{pmatrix}^{-1}$*?*

Using the adjoint formula for an inverse, we know that it is

$$\frac{-\begin{vmatrix} 1 & 2 & 5 \\ 1 & 2 & 4 \\ 2 & 0 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & -3 & 2 & 5 \\ 1 & -3 & 2 & 4 \\ 0 & 1 & -3 & 2 \\ 2 & -1 & 0 & 3 \end{vmatrix}} = -\frac{6 + 16 + 0 - 0 - 6 - 20}{-11} = \frac{-4}{11}.$$

7. **(10 points)** *For each of the following subsets S of a named vector space V, explain whether S is or is not a subspace of V and why.*

   (a) $V = \mathbb{R}^{3\times 3}$, $S = \{A : \det A = 0\}$.
   Here S is not a subspace, as the sum of two singular matrices could be nonsingular, e.g. $\begin{pmatrix} 1&0&0\\0&1&0\\0&0&0 \end{pmatrix} + \begin{pmatrix} 0&0&0\\0&0&0\\0&0&1 \end{pmatrix} = \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}$.

   (b) $V = \mathbb{R}^4$, $S = \{(w,x,y,z)^T : w+x+y+z=0\}$.
   Here S is a subspace, and in particular it's actually the nullspace of the matrix $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$. To show from more general principles that it's a subspace, note that for any two vectors $(w_1,x_1,y_1,z_1)^T$ and $(w_2,x_2,y_2,z_2)^T$ satisfying this required equation, it is also true that $(w_1+w_2)+(x_1+x_2)+(y_1+y_2)+(z_1+z_2)=0$, and that for any real k, $(kw_1)+(kx_1)+(ky_1)+(kz_1)=0$.

   (c) $V = P_5$, $S = \{f(x) : \text{the degree of } f \text{ is less than } 3\}$.
   Here S is a subspace, and it's specifically P_2. Note that the sum of two polynomials of degree less than 3 is itself a polynomial of degree less than 3, and likewise the product of a polynomial with a real number either leaves its degree unchanged or (if the real number is zero) turns it into a polynomial of degree zero.

8. **(15 points)** *Answer the following questions related to the matrix* $A = \begin{pmatrix} 1 & 2 & -3 & -1 \\ -2 & -1 & 5 & 2 \\ -1 & 4 & 1 & 1 \end{pmatrix}$.

   (a) **(10 points)** *Calculate the dimension of the nullspace of A, and find a basis.*
   Using Gauss-Jordan elimination on A (which leaves its nullspace unchanged):

$$\begin{pmatrix} 1 & 2 & -3 & -1 \\ -2 & -1 & 5 & 2 \\ -1 & 4 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -3 & -1 \\ 0 & 3 & -1 & 0 \\ 0 & 6 & -2 & 0 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 2 & -3 & -1 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 6 & -2 & 0 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 0 & -\frac{7}{3} & -1 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

   The nullspace has dimension 2, since the above matrix have two pivotless columns and thus two free variables. Specifically, we see that elements of the nullspace have the form $(\frac{7}{3}x_3 + x_4, \frac{1}{3}x_3, x_3, x_4)^T$. The set of such vectors is spanned by the basis

$$\left\{ \begin{pmatrix} \frac{7}{3} \\ \frac{1}{3} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \text{ or } \left\{ \begin{pmatrix} 7 \\ 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

   where the second form is simply a more attractive presentation produced by scaling one of the basis vectors.

(b) **(10 points)** *Is the set of vectors* $\left\{ \begin{pmatrix} 1 \\ 2 \\ -3 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 5 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \\ 1 \\ 1 \end{pmatrix} \right\}$ *linearly independent? Explain your reasoning.*

Since the Gauss-Jordan process on a matrix with these vectors as rows eventually produced a zero row, we know that these vectors are not linearly independent. More specifically, we know that these three vectors span a space also spanned by the two vectors

$$\begin{pmatrix} 1 \\ 0 \\ -\frac{7}{3} \\ -1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \\ -\frac{1}{3} \\ 0 \end{pmatrix}$$

and since two vectors suffice to span this space, any three vectors in this space must not be linearly independent.

A purely arithmetic argument could have been determined by noting exactly how the zero row in the above Gauss-Jordan elimination appears. Specifically,

$$-3 \begin{pmatrix} 1 \\ 2 \\ -3 \\ -1 \end{pmatrix} - 2 \begin{pmatrix} -2 \\ -1 \\ 5 \\ 2 \end{pmatrix} + 1 \begin{pmatrix} -1 \\ 4 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

but this specific example is not necessary to show linear dependence.