This test is closed-book and closed-notes. For full credit show all of your work (legibly!), unless otherwise specified.

1. **(10 points)** Determine whether the following system is consistent or inconsistent; if it is consistent, find all solutions.

$$\begin{cases} x_1 + 3x_2 + x_3 + x_4 = 3 \\ 2x_1 - 2x_2 + x_3 + 2x_4 = 8 \\ x_1 - 5x_2 \qquad + x_4 = 5 \end{cases}$$

2. **(10 points)** Let $A = \begin{pmatrix} 4 & 1 & 6 \\ 2 & 3 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 3 & 0 \\ -2 & 2 & -4 \end{pmatrix}$, and $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$. For each of the following arithmetic expressions, either calculate its value or explain briefly why it cannot be calculated.

   (a) $(A + B)\mathbf{v}$.

   (b) AB.

   (c) AB^T.

   (d) $A^T B\mathbf{v}$.

3. **(10 points)** Using any method you like, find the inverse of the matrix $\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & -2 & -3 \end{pmatrix}$.

4. **(5 points)** Calculate the determinant $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 7 & 3 \end{vmatrix}$.

5. **(5 points)** Determine a value of k such that the matrix $\begin{pmatrix} 1 & 2 & 3 \\ k & 1 & k \\ 5 & 2 & 4 \end{pmatrix}$ is singular.

6. **(15 points)** For each of the vector spaces U and subsets V, determine, with justification, whether V is a *subspace* of U.

   (a) $U = P_3$, and V is the set of polynomials $f(x)$ in U such that $f(3) = 0$.

   (b) $U = \mathbb{R}^4$, and V is the set of vectors $(w, x, y, z)^{\mathrm{T}}$ such that $w + x + y + z = 1$.

   (c) $U = C[0, 1]$ (the set of continuous functions on $[0, 1]$), and V is the set of differentiable functions in U.

7. **(10 points)** Determine (with explanation) whether the set of vectors $\left\{\begin{pmatrix}2\\1\\-2\end{pmatrix}, \begin{pmatrix}3\\2\\-2\end{pmatrix}, \begin{pmatrix}2\\2\\0\end{pmatrix}\right\}$ in $\mathbb{R}^3$ is linearly independent.

8. **(15 points)** The matrix $A = \begin{pmatrix}2 & 0 & 3 & -4 & -28\\-4 & 0 & 1 & -2 & -22\\3 & 0 & -7 & 1 & 39\end{pmatrix}$ is row-equivalent to the reduced row-echelon form $\begin{pmatrix}1 & 0 & 0 & 0 & 2\\0 & 0 & 1 & 0 & -4\\0 & 0 & 0 & 1 & 5\end{pmatrix}$. Answer the following questions.

   (a) What is the rank of A?

   (b) What is the dimension of $R(A)$, and a basis for $R(A)$?

   (c) What is the dimension of $R(A^{\mathrm{T}})$, and a basis for $R(A^{\mathrm{T}})$?

   (d) What is the dimension of $N(A)$, and a basis for $N(A)$?

9. **(15 points)** Answer the following questions about linear transformations.

   (a) On the vector space P_3, is $L(f(x)) = f(3)f'(x)$ a linear transformation? If not, why not; if so, what matrix represents it with respect to the standard basis $[1, x, x^2, x^3]$?

   (b) On the vector space $C[0, 1]$, is $L(f(x)) = f(1)x + f(0)$ a linear transformation? If not, why not; if so, why so?

   (c) On the vector space $\mathbb{R}^3$, is $L((x, y, z)^{\mathrm{T}}) = (x^2 - y^2, x^2 + y^2, z)^{\mathrm{T}}$ a linear transformation? If not, why not; if so, what matrix represents it with regard to the standard basis $[(1, 0, 0)^{\mathrm{T}}, (0, 1, 0)^{\mathrm{T}}, (0, 0, 1)^{\mathrm{T}}]$?

10. **(10 points)** Determine the distance from the point $(2, 3)$ to the line $x + 4y = 8$.

11. **(10 points)** Using the inner product $\langle f(x), g(x)\rangle = \int_0^1 f(x)g(x)dx$ on the vector space $C[0,1]$, find the cosine of the angle between e^x and e^{-x}.

12. **(10 points)** Find an orthonormal basis for the space spanned by $(5,-12,0)^{\mathrm{T}}$ and $(1,1,1)^{\mathrm{T}}$.

13. **(10 points)** Determine the eigenvectors and eigenvalues of $\begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ 3 & 6 & -3 \end{pmatrix}$.