1. **(6 points)** *For each of the following pairs of matrices A and B, identify the* elementary *matrix E such that $B = EA$.*

   (a) **(2 points)** $A = \begin{pmatrix} 5 & 2 & -3 \\ 2 & -6 & 0 \\ 8 & 3 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 5 & 2 & -3 \\ 1 & -3 & 0 \\ 8 & 3 & 1 \end{pmatrix}$.

   B is the result of scaling the second row of A by a factor of $\frac{1}{2}$. This is a Type II elementary row operation, which can be represented by a left multiplication by the elementary matrix $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

   (b) **(2 points)** $A = \begin{pmatrix} 1 & -1 & 6 \\ 3 & 5 & 2 \\ 4 & 1 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 & 6 \\ 3 & 5 & 2 \\ 0 & 5 & -19 \end{pmatrix}$.

   B is the result of subtracting four times the first row from the third row of A. This is a Type III elementary row operation, which can be represented by a left multiplication by the elementary matrix $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix}$.

   (c) **(2 points)** $A = \begin{pmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 4 & 3 \\ 2 & 2 & 3 \\ -1 & -2 & 0 \end{pmatrix}$.

   B is the result of swapping the second and third rows of A. This is a Type I elementary row operation, which can be represented by a left multiplication by the elementary matrix $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

2. **(9 points)** *Answer the following questions.*

   (a) **(6 points)** *Using any technique you like, calculate the inverse of the matrix* $A = \begin{pmatrix} -1 & -3 & -3 \\ 2 & 6 & 1 \\ 3 & 8 & 3 \end{pmatrix}$.

   There are several sensible ways to solve this problem; two of them are below.

   One approach is to use Gauss-Jordan elimination to convert the augmented matrix $\left(A \mid I \right)$ to $\left(I \mid A^{-1} \right)$. This process is shown below:

$$\left(\begin{array}{ccc|ccc} -1 & -3 & -3 & 1 & 0 & 0 \\ 2 & 6 & 1 & 0 & 1 & 0 \\ 3 & 8 & 3 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 3 & 3 & -1 & 0 & 0 \\ 0 & 0 & -5 & 2 & 1 & 0 \\ 0 & -1 & -6 & 3 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 3 & 3 & -1 & 0 & 0 \\ 0 & 1 & 6 & -3 & 0 & -1 \\ 0 & 0 & -5 & 2 & 1 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & -15 & 8 & 0 & 3 \\ 0 & 1 & 6 & -3 & 0 & -1 \\ 0 & 0 & -5 & 2 & 1 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & -15 & 8 & 0 & 3 \\ 0 & 1 & 6 & -3 & 0 & -1 \\ 0 & 0 & 1 & -0.4 & -0.2 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -3 & 3 \\ 0 & 1 & 0 & -0.6 & 1.2 & -1 \\ 0 & 0 & 1 & -0.4 & -0.2 & 0 \end{array}\right)$$

   so $A^{-1} = \begin{pmatrix} 2 & -3 & 3 \\ -0.6 & 1.2 & -1 \\ -0.4 & -0.2 & 0 \end{pmatrix}$.

One could also arrive at this result by using the adjoint-and-determinant calculation of the inverse:

$$
\begin{aligned}
A^{-1} &= \frac{1}{\det A}\begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} \\
&= \frac{1}{\begin{vmatrix} -1 & -3 & -3 \\ 2 & 6 & 1 \\ 3 & 8 & 3 \end{vmatrix}} \begin{pmatrix} \begin{vmatrix} 6 & 1 \\ 8 & 3 \end{vmatrix} & -\begin{vmatrix} -3 & -3 \\ 8 & 3 \end{vmatrix} & \begin{vmatrix} -3 & -3 \\ 6 & 1 \end{vmatrix} \\ -\begin{vmatrix} 2 & 1 \\ 3 & 3 \end{vmatrix} & \begin{vmatrix} -1 & -3 \\ 3 & 3 \end{vmatrix} & -\begin{vmatrix} -1 & -3 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & 6 \\ 3 & 8 \end{vmatrix} & -\begin{vmatrix} -1 & -3 \\ 3 & 8 \end{vmatrix} & \begin{vmatrix} -1 & -3 \\ 2 & 6 \end{vmatrix} \end{pmatrix} \\
&= \frac{1}{5}\begin{pmatrix} 10 & -15 & 15 \\ -3 & 6 & -5 \\ -2 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & -3 & 3 \\ -0.6 & 1.2 & -1 \\ -0.4 & -0.2 & 0 \end{pmatrix}
\end{aligned}
$$

(b) **(3 points)** *Making use of the result above, determine the unique solution to the matrix equation* $\left(\begin{smallmatrix} -1 & -3 & -3 \\ 2 & 6 & 1 \\ 3 & 8 & 3 \end{smallmatrix}\right)\mathbf{x} = \left(\begin{smallmatrix} 3 \\ 0 \\ -2 \end{smallmatrix}\right)$.

When A is nonsingular, the solution to the matrix equation $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = A^{-1}\mathbf{b}$, so here

$$
\mathbf{x} = \begin{pmatrix} 2 & -3 & 3 \\ -0.6 & 1.2 & -1 \\ -0.4 & -0.2 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0.2 \\ -1.2 \end{pmatrix}
$$

3. **(5 points)** *Using any technique you like, calculate the determinant* $\begin{vmatrix} 1 & 4 & 0 & 2 \\ 2 & -5 & 0 & 4 \\ 6 & -3 & 3 & 8 \\ 0 & 1 & 0 & 2 \end{vmatrix}$.

The most straightforward way to do this is by cofactor expansion on the third column (since only one entry in this column is nonzero. The determinant would then be a sum of four cofactors, three of which are zero, and the last of which is $3 \cdot \begin{vmatrix} 1 & 4 & 2 \\ 2 & -5 & 4 \\ 0 & 1 & 2 \end{vmatrix}$. The determinant of that matrix can be calculated straightofrwardly to be $-10 + 0 + 4 - 4 - 16 - 0 = -26$, so the determinant of the original matrix is $3(-26) = -78$.

Alternatively, one could row-reduce the matrix, keeping track of the factors introduced by type I and type II row operations:

$$
\begin{vmatrix} 1 & 4 & 0 & 2 \\ 2 & -5 & 0 & 4 \\ 6 & -3 & 3 & 8 \\ 0 & 1 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 0 & 2 \\ 0 & -13 & 0 & 0 \\ 0 & -27 & 3 & -4 \\ 0 & 1 & 0 & 2 \end{vmatrix} = (-13)\begin{vmatrix} 1 & 4 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & -27 & 3 & -4 \\ 0 & 1 & 0 & 2 \end{vmatrix} = (-13)\begin{vmatrix} 1 & 4 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & 2 \end{vmatrix}
$$

and since the last determinant is of an upper triangular matrix, we can compute its determinant by multiplying along the diagonal to get $-13 \cdot 1 \cdot 1 \cdot 3 \cdot 2 = -78$.