1. **(6 points)** *For each of the following yes/no questions, include a brief justification of your answer.*

   (a) **(3 points)** *Is the set of vectors* $\{(x, y, z)^T | x + y + z = 1\}$ *a subspace of* $\mathbb{R}^3$*? Why or why not?*

   It is not, because it lacks the closure properties: both $(1, 0, 0)^T$ and $(0, 1, 0)^T$ are in this set, but their sum $(1, 1, 0)^T$ is not (many other equally good counterexamples exist).

   (b) **(3 points)** *Is the set of polynomials* f *of degree 4 or less such that* $f(7) = 0$ *a subspace of* P_4*? Why or why not?*

   It is a subspace, because it is closed under addition and scalar multiplication. If f and g are both in this set (so $f(7) = g(7) = 0$), then since $(f + g)(x) = f(x) + g(x)$, it follows that $(f + g)(7) = f(7) + g(7) = 0 + 0 = 0$ and so $f + g$ is in this set as well; likewise, for any real number k, $(kf)(7) = k \cdot f(7) = k \cdot 0 = 0$, so kf is also an element of this set.

2. **(11 points)** *Answer the following related questions.*

   (a) **(6 points)** *Find a basis for the nullspace of* $\begin{pmatrix} 1 & 3 & -4 & 4 \\ 2 & 0 & 1 & -1 \\ 4 & 1 & 2 & 4 \end{pmatrix}$.

   A sensible way to start is to perform Gauss-Jordan elimination on this matrix, since elementary row operations do not change the nullspace.

$$\begin{pmatrix} 1 & 3 & -4 & 4 \\ 2 & 0 & 1 & -1 \\ 4 & 1 & 2 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & -4 & 4 \\ 0 & -6 & 9 & -9 \\ 0 & -11 & 18 & -12 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 3 & -4 & 4 \\ 0 & 1 & -\frac{3}{2} & \frac{3}{2} \\ 0 & -11 & 18 & -12 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -\frac{3}{2} & \frac{3}{2} \\ 0 & 0 & \frac{3}{2} & \frac{9}{2} \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -\frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 1 & 3 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

   Since we want to know which vectors $(w, x, y, z)^T$, when multiplied on the left with this matrix, become the zero vector, we require that $w - 2z = 0$, $x + 6z = 0$, and $y + 3z = 0$. In other words, $w = 2z$, $x = -6z$, and $y = -3z$, so that elements of the nullspace have the form $(2z, -6z, -3z, z)^T$. This is clearly an aribtrary linear combination of the single vector $(2, -6, -3, 1)^T$, so that the above nullspace has, as a basis, the one-element set $\left\{ \begin{pmatrix} 2 \\ -6 \\ -3 \\ 1 \end{pmatrix} \right\}$.

(b) **(2 points)** *What is the dimension of the nullspace above, and what common named vector space is it a subspace of?*

It is clearly a subspace of $\mathbb{R}^4$, since its elements are 4-dimensional vectors. This space itself, however, has a basis of a single element, so it is specifically a one-dimensional subspace of $\mathbb{R}^4$.

(c) **(3 points)** *Is the set of vectors* $\left\{ \begin{pmatrix} 1 \\ 3 \\ -4 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \\ 2 \\ 4 \end{pmatrix} \right\}$ *linearly independent? Why or why not?.*

From the above Gauss-Jordan process we may see that they are linearly independent, since the elementary row operations, which do not change linear independence of the rows, resulted in three clearly independent rows since each row has a pivot.

3. **(3 points)** *Considered as elements of the vector space of all polynomials on x, show that the set of polynomials $\{x(x-1)(x-2), x(x-1)(x-3), x(x-2)(x-3), (x-1)(x-2)(x-3)\}$ is linearly independent. (Hint: the zero polynomial is the only one which evaluates to zero for each input value x)*

Suppose $c_1x(x-1)(x-2) + c_2x(x-1)(x-3) + c_3x(x-2)(x-3) + c_4(x-1)(x-2)(x-3) \equiv 0$. Evaluating this at $x = 0$, we find that $-6c_4 = 0$, so $c_4 = 0$. Evaluating it at $x = 1$, we find that $2c_3 = 0$ so $c_3 = 0$. Likewise,w hen $x = 2$ and $x = 3$, we can ascertain that $-2c_2 = 0$ and $6c_1 = 0$ respectively, so $c_1 = c_2 = c_3 = c_4 = 0$. Since this is the unique solution to the above linear equation, the participating vectors are linearly independent.