1. **(4 points)** *Find a basis for the column space of the matrix* $A = \begin{pmatrix} 1 & 2 & -3 & 2 \\ 0 & 3 & 4 & -5 \\ 3 & 3 & -13 & 11 \end{pmatrix}$.

   If we place the matrix A in row-echelon form, we find that

$$A = \begin{pmatrix} 1 & 2 & -3 & 2 \\ 0 & 3 & 4 & -5 \\ 3 & 3 & -13 & 11 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -3 & 2 \\ 0 & 3 & 4 & -5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

   which has rank 2 and pivots in the first and second columns; thus the first and second columns of A are linearly independent but the remaining columns are dependent on them (you might note specifically that the third column is the sum of $\frac{-17}{3}$ times the first column and $\frac{4}{3}$ times the second column, while the fourth column is the sum of $\frac{16}{3}$ times the first column and $\frac{-5}{3}$ times the second column, if you want an explicit argument for this fact). Thus, the column-space of A is spanned by the two linearly independent vectors $(1, 0, 3)^{\mathrm{T}}$ and $(2, 3, 3)^{\mathrm{T}}$, which thus form a basis. Naturally, other bases are also possible, as this particular two-dimensional space has a wide range of vectors which span it, but this basis is the most straightforward one to select.

2. **(6 points)** *For each of the following maps L from one vector space to another, determine whether L is a linear transformation or not, and state your reasoning.*

   (a) $L : P_3 \to \mathbb{R}^2$ *where* $L(f(x)) = \begin{pmatrix} f(0) \\ f(3) \end{pmatrix}$.

   This is a linear transformation. One way to demonstrate this is to note that for any polynomials $f(x)$ and $g(x)$ and scalar k,

$$L(f+g) = \begin{pmatrix} (f+g)(0) \\ (f+g)(3) \end{pmatrix} = \begin{pmatrix} f(0)+g(0) \\ f(3)+g(3) \end{pmatrix} = \begin{pmatrix} f(0) \\ f(3) \end{pmatrix} + \begin{pmatrix} g(0) \\ g(3) \end{pmatrix} = L(f) + L(g)$$

$$L(kf) = \begin{pmatrix} (kf)(0) \\ (f)(3) \end{pmatrix} = \begin{pmatrix} kf(0) \\ kf(3) \end{pmatrix} = k\left(f(0) f(3)\right) = kL(f)$$

   Alternatively, one could note that, if P_3 is coordinatized by the standard basis, L is represented by the 2×4 matrix $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 9 & 27 \end{pmatrix}$.

   (b) $L : \mathbb{R}^2 \to \mathbb{R}^4$ *where* $L\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x \\ 2x \\ 0 \\ 3x \end{pmatrix}$.

   This is a linear transformation. Probably the easiest way to demonstrate this is by noting that L transforms its parameter specifically by matrix multiplication by the 4×2 matrix $A = \begin{pmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 0 \\ 3 & 0 \end{pmatrix}$ and so $L(\vec{u} + \vec{v}) = A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v} = L(\vec{u}) + L(\vec{v})$ and likewise $L(k\vec{u}) = A(k\vec{u}) = k(A\vec{u}) = kL(\vec{u})$.

   (c) $L : \mathbb{R}^3 \to \mathbb{P}_3$ *where* $L\left(\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}\right) = (x - r_1)(x - r_2)(x - r_3)$.

   This is not a linear transformation. One easy way to see this is because $L(\vec{0}) = x^3$, but $L(k\vec{0}) = L(\vec{0}) \neq kx^3$ for values of $k \neq 1$.

3. **(4 points)** *Determine (with brief explanation) the kernel and range of the linear transformation on P_4 given by $L(f(x)) = f''(x)$.*

The kernel of L is the set of all polynomials whose second derivative is identically zero; that is to say, all linear polynomials or elements of P_1. Similarly, the range of L is the set of possible second derivatives of polynomials of degree 4 or less; this set is identically the set of all polynomials of degree 2 or less, also known as P_2.

4. **(6 points)** *Let L be a linear transformation on P_2 given by $L(f(x)) = xf'(x) + f(2)$. Find a matrix A representing L with respect to the standard basis $[1, x, x^2]$.*

   Note $L(1) = x \cdot 0 + 1 = 1 + 0x + 0x^2$, $L(x) = x \cdot 1 + 2 = 2 + x + 0x^2$, and $L(x^2) = x \cdot 2x + 4 = 4 + 0x + 2x^2$. Thus, building column vectors of the coefficients from the images of these three basis elements, we find the matrix associated with this transformation to be $\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.