1. **(6 points)** *Let L be a linear transformation on $\mathbb{R}^2$ given by $L(\mathbf{x}) = (x_1 - x_2, 2x_1 + x_2)^{\mathrm{T}}$.*

   (a) **(2 points)** *What is the matrix M representing L with respect to the standard basis $[\mathbf{e}_1, \mathbf{e}_2]$?*
   Since $L(\mathbf{e}_1) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \mathbf{e}_1 + 2\mathbf{e}_2$ and $L(\mathbf{e}_2) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -\mathbf{e}_1 + \mathbf{e}_2$, the matrix representation of L is $M = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$.

   (b) **(4 points)** *What is the matrix N representing L with respect to the nonstandard basis $[\mathbf{u}_1, \mathbf{u}_2]$, where $\mathbf{u}_1 = (1,1)^{\mathrm{T}}$ and $\mathbf{u}_2 = (-1,0)^{\mathrm{T}}$?*
   This question can be approached in either of two ways: by implementing a change-of-basis similarity between N and M, or working out N from first principles.
   To work out N from first principles, we see how $L(\mathbf{u}_1)$ and $L(\mathbf{u}_2)$ can be written as linear combinations of the two vectors:

$$L(\mathbf{u}_1) = \begin{pmatrix} 0 \\ 3 \end{pmatrix} = 3\mathbf{u}_1 + 3\mathbf{u}_2$$

$$L(\mathbf{u}_2) = \begin{pmatrix} -1 \\ -2 \end{pmatrix} = -2\mathbf{u}_1 - \mathbf{u}_2$$

   so $N = \begin{pmatrix} 3 & -2 \\ 3 & -1 \end{pmatrix}$.
   Alternatively, we could implement a change-of-basis relationship $N = S^{-1}MS$, where S is the matrix whose columns are $\mathbf{u}_1$ and $\mathbf{u}_2$, in that order. Then:

$$\begin{aligned} N &= \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 3 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 3 & -2 \\ 3 & -1 \end{pmatrix}. \end{aligned}$$

2. **(4 points)** *What is the closest point to $(1,3,6)$ on the plane $2x - 3y + 3z = 0$?*
   The plane has a normal vector $\mathbf{n} = (2,-3,3)^{\mathrm{T}}$, and the given point can be represented by a vector $\mathbf{u} = (1,3,6)^{\mathrm{T}}$. We wish to decompose $\mathbf{u}$ into a projection $\mathbf{p}$ onto $\mathbf{n}$ and a component $\mathbf{z}$ orthogonal to $\mathbf{n}$, which will lie in the plane $2x - 3y + z$. Using the known vector projection formula,

$$\mathbf{p} = \mathrm{proj}_{\mathbf{n}}\, \mathbf{u} = \frac{\mathbf{u}^{\mathrm{T}}\mathbf{n}}{\mathbf{n}^{\mathrm{T}}\mathbf{n}}\mathbf{n} = \frac{11}{22}\mathbf{n} = \begin{pmatrix} 1 \\ -\frac{3}{2} \\ \frac{3}{2} \end{pmatrix}$$

   , and since $\mathbf{p} + \mathbf{z}$ should be $\mathbf{u}$, we get

$$\mathbf{z} = \mathbf{u} - \mathbf{p} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ -\frac{3}{2} \\ \frac{3}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{9}{2} \\ \frac{9}{2} \end{pmatrix},$$

   so the point in question is $(0, 4.5, 4.5)$.

3. **(3 points)** *What is the closest point to* $(2,0,0)$ *on the line which passes through both* $(0,0,0)$ *and* $(-1,3,4)$*?*

The line in question can be represented by the vector $\mathbf{v} = (-1,3,4)^{\mathrm{T}}$, while the point given is represented by $\mathbf{u} = (2,0,0)$. The nearest point on the line to the given point will just be

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u}^{\mathrm{T}}\mathbf{v}}{\mathbf{v}^{\mathrm{T}}\mathbf{v}}\mathbf{v} = \frac{-2}{26}\mathbf{v} = \begin{pmatrix} \frac{1}{13} \\ \frac{-3}{13} \\ \frac{-4}{13} \end{pmatrix},$$

so the nearest point to $(2,0,0)$ on the given line is $(\frac{1}{13}, \frac{-3}{13}, \frac{-4}{13})$.

4. **(7 points)** *Let* U *be the subspace of* $\mathbb{R}^4$ *spanned by the vectors* $(1,3,2,0)^{\mathrm{T}}$, $(2,5,0,2)^{\mathrm{T}}$, *and* $(4,11,4,2)^{\mathrm{T}}$. *Find a basis for* $U^{\perp}$ *and state its dimension.*

Note that U is $R(A^T)$, where $A = \begin{pmatrix} 1 & 3 & 2 & 0 \\ 2 & 5 & 0 & 2 \\ 4 & 11 & 4 & 2 \end{pmatrix}$. Then we know $U^{\perp}$ to be $N(A)$. Finding a nullspace is most easily done by row-reducing the matrix under consideration, so we proceed to do so:

$$\begin{aligned} \begin{pmatrix} 1 & 3 & 2 & 0 \\ 2 & 5 & 0 & 2 \\ 4 & 11 & 4 & 2 \end{pmatrix} &\sim \begin{pmatrix} 1 & 3 & 2 & 0 \\ 0 & -1 & -4 & 2 \\ 0 & -1 & -4 & 2 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & 3 & 2 & 0 \\ 0 & 1 & 4 & -2 \\ 0 & -1 & -4 & 2 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & 0 & -10 & 6 \\ 0 & 1 & 4 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

so elements of the nullspace satisfy $x_1 = 10x_3 - 6x_4$ and $x_2 = -2x_3 + 2x_4$, or in other words, elements of the nullspace have the form

$$\begin{pmatrix} 10x_3 - 6x_4 \\ -2x_3 + 2x_4 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 10 \\ -2 \\ 1 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} -6 \\ 2 \\ 0 \\ 1 \end{pmatrix} x_4$$

yielding the two given vectors as the basis for the nullspace, which has dimension 2.