1. Each of the following statements is one which can be proven by contradiction (other approaches, like proof by contrapositive, might also work). We'll be constructing contradiction proofs of these statements in class.

   (a) For any real numbers x and y, if $x + y$ is irrational, then x or y is irrational.

   (b) For any integers a, b, and c such that $a^2 + b^2 = c^2$, it is the case that either a or b is even.

   (c) For any natural number $n > 1$, if a is the smallest natural number greater than n which divides n, then a is prime.

2. Let's look at each of the following proofs and see if they seem watertight; evaluate them for logical flaws or flaws of construction.

   (a) For all real numbers x and y, if x is irrational and y is rational, then $x + y$ is irrational.

   Proof. By way of contradiction, let us assume that the proposition is false, so that there exist real numbers x and y such that $x \notin \mathbb{Q}$, $y \in \mathbb{Q}$, and $x + y \in \mathbb{Q}$. Since the rational numbers are closed under subtraction and both $x + y$ and y are rational, we know that

   $$(x + y) - y \in \mathbb{Q}.$$

   Simplifying the above yields that $x \in \mathbb{Q}$, which contradicts our assumption that $x \notin \mathbb{Q}$. Thus our proposition is not false. □

   (b) For each real number x, $x(1 - x) \leq \frac{1}{4}$.

   Proof. We shall prove this by contradiction, so let us counterfactually assume that there is a real number x such that $x(1 - x) > \frac{1}{4}$. Multiplying both sides of this inequality by 4 gives $4x(1 - x) > 1$. However, letting $x = 3$, we see that

   $$4 \cdot 3(1 - 3) > 1$$

   which is the false statement $-12 > 1$. This is a contradiction and so our original proposition is proven. □

   (c) For each real number x, if x is irrational and m is an integer, then mx is irrational.

   Proof. We know by our premise that x is a real irrational number, so for all integers a and b with $b \neq 0$, $x \neq \frac{a}{b}$. Thus we may conclude that $mx \neq \frac{ma}{b}$ and so mx is irrational. □