1. **(11 points)** *The statement "for any real number x, if x is not an integer, then $7x$ is also not an integer" is false. State its converse. Is the converse true or not? Briefly explain your reasoning.*

   The converse of this statement is "for any real number x, if $7x$ is not an integer, then x is not an integer." This is a true statement; one easy way to show it is to look at its contrapositive: if x is an integer, then $7x$ is surely an integer.

2. **(18 points)** *Fill in the truth table for each of the following statements, and identify the statement as a tautology, a contradiction, or neither.*

   (a) $(P \vee Q) \leftrightarrow P$.

P	Q	$P \vee Q$	$(P \vee Q) \leftrightarrow P$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	T

   This statement is neither a contradiction nor a tautology.

   (b) $[(P \to Q) \wedge \sim Q] \to \sim P$.

P	Q	$P \to Q$	$\sim Q$	$(P \to Q) \wedge \sim Q$	$\sim P$	$[(P \to Q) \wedge \sim Q] \to \sim P$
T	T	T	F	F	F	T
T	F	F	T	F	F	T
F	T	T	F	F	T	T
F	F	T	T	T	T	T

   Since this statement is always true, it is a tautology.

3. **(22 points)** *Prove that for any real number x, if $x^3 + x < 10$, then $x < 2$.*

   Because this statement has a cumbersome premise and a simple conclusion, recasting it in terms of its contrapositive shows promise.

   Proof. Let us consider instead the contrapositive statement that if $x \geq 2$, then $x^3 + x \geq 10$. From our new premise, since $x \geq 2$, $x^3 \geq 8$, and thus $x^3 + x \geq 8 + 2 = 10$. □

4. **(22 points)** *Prove that for any integers m and n, if n is odd, then $nm - m^2$ is even.*

   Since parity is clearly relevant to our conclusion but we are not given any guidance in the premise as to whether m is even or odd, we shall divide up into cases based on the parity of m.

   Proof. We may note from our premise that since n is odd, there is an integer k such that $n = 2k + 1$. Then we consider two separate cases, depending on the parity of m.

   Case I: m is even. Then by our case premise $m = 2\ell$ for some integer ℓ. Then,

$$nm - m^2 = (2k + 1)2\ell - (2\ell)^2 = 2(2k\ell + \ell - 2\ell^2)$$

   demonstrating that the above expression is even.

   Case I: m is odd. In this case, our case premise informs us that $m = 2\ell + 1$ for some integer ℓ. Then,

$$nm - m^2 = (2k+1)(2\ell+1) - (2\ell+1)^2 = 4k\ell + 2k + 2\ell + 1 - (4\ell^2 + 4\ell + 1) = 2(2k\ell + k - \ell - 2\ell^2)$$

and since $k\ell + k - \ell - 2\ell^2$ is an integer, the above expression is once again even.

$\square$

5. **(20 points)** *Prove that for any positive integer n, it is the case that*

$$1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + 4 \cdot 2^4 + \cdots + n \cdot 2^n = (n-1)2^{n+1} + 2$$

Proof. We proceed by induction on n; note that the base case $n = 1$ can be demonstrated by arithmetic: $1 \cdot 2^1 = 2 = (0)2^2 + 2$. Now, we assume, for a specific n, that it is true that

$$1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + 4 \cdot 2^4 + \cdots + n \cdot 2^n = (n-1)2^{n+1} + 2$$

and seek to prove that

$$1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + 4 \cdot 2^4 + \cdots + n \cdot 2^n + (n+1) \cdot 2^{n+1} = n2^{n+2} + 2$$

To do so, we may add $(n+1)2^{n+1}$ to both sides of our inductive hypothesis, and proceed via arithmetic:

$$\begin{aligned} 1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + 4 \cdot 2^4 + \cdots + n \cdot 2^n + (n+1)2^{n+1} &= (n-1)2^{n+1} + 2 + (n+1)2^{n+1} \\ &= (2n)2^{n+1} + 2 \\ &= n2^{n+2} + 2 \end{aligned}$$

which is the desired result. $\square$

6. **(20 points)** *Prove that there is no integer n such that $n \equiv 4 \pmod 6$ and $n \equiv 2 \pmod 9$.*

Proof. Let us assume counterfactually that there is such an n. Then $6 \mid n - 4$ and $9 \mid n - 2$, so there are integers k and ℓ such that $n - 4 = 6k$ and $n - 2 = 9\ell$. Subtracting these two equations from each other, we get $-2 = 6k - 9\ell = 3(2k - 3\ell)$. Since $2k - 3\ell$ is an integer, we then conclude that $3 \mid -2$, which contradicts the known arithmetic fact that $3 \nmid -2$. $\square$