1. (8 points) Calculate the following integrals:

(a) (4 points) $\int_{0}^{\pi/2} (\sin \theta \cos^{3} \theta) d\theta$.

The substitution in this integral is suggested by two notable features: first, the existence of the composition $(\cos \theta)^3$ in the integrand, and the fact that the integrand involves both a particular function $(\cos \theta)$ and its derivative $(-\sin \theta)$. Based on both these hints it seems that the substitution $u = \cos \theta$ would be best for solving this problem. Then $du = (d/d\theta \cos \theta) \, dx = -\sin \theta \, d\theta$, so the phrase $\sin \theta \, d\theta$ appearing in the integral above can be rewritten as $(-du)$. Thus the above integral becomes

$$\int_{u=0}^{u=\cos \pi/2} u^3 (-du) = \left[ -u^4/4 \right]_{0}^{\cos \pi/2} = \left[ -\cos^4 \theta/4 \right]_{0}^{\pi/2} = -\cos^4 \pi/2 + \cos^4 0 = 1/4$$

(b) (4 points) $\int 5e^{3t} + 2dt$.

We start by separating the sum into two distinct integrals, since 2 is easily integrated: $\int 5e^{3t} + 2dt = \int 5e^{3t} \, dt + 2t$. Then, the remaining term can be solved either with an explicit substitution $u = 3t$ or the implicit rule for linear substitution; in either case, a $1/3$ term is introduced, so the result is $\frac{5}{3}e^t + 2t + C$.

2. (8 points) Calculate the area in the region between $y = \frac{2x}{\pi}$ and $y = \sin x$ shown below:

As shown on the graph, this region begins on the left at $x = 0$ and ends on the right at $x = \frac{\pi}{2}$. Furthermore, $y = \sin x$ is the higher of the two functions throughout its length, so all necessary information to construct the integral is on hand:

$$\int_{0}^{\pi/2} \sin x - \frac{2x}{\pi} \, dx = \left[ -\cos x - \frac{1}{\pi} x^2 \right]_{0}^{\pi/2} = (-0 - \frac{1}{\pi} \left( \frac{\pi}{2} \right)^2) - (-1 - 0) = 1 - \frac{\pi}{4}$$

3. (8 points) Calculate the volume of the solid produced by rotating the region bounded by the $y$-axis, $y = \frac{12}{x}$, $y = 2$, and $y = 3$ around the $y$-axis.
Using the disc method with respect to the \( y \)-axis, we’ll need to integrate with respect to \( y \) (i.e., working bottom to top). Since our boundaries are \( y = 2 \) and \( y = 3 \), we will integrate from 2 to 3. At a particular height \( y \), the right boundary is \( y = \frac{12}{y} \), which gives us \( x = \frac{12}{y} \), so the radius of each slice is \( \frac{12}{y} \). Thus, our integral is:

\[
\int_{2}^{3} \pi \left( \frac{12}{y} \right)^2 dy = \int_{2}^{3} 144\pi y^{-2} = -144\pi y^{-1}\bigg|_{2}^{3} = -\frac{144}{3}\pi + \frac{144}{2}\pi = (72 - 48)\pi = 24\pi
\]

It is possible, but actually considerably more difficult, to perform this integral with respect to \( x \) and use the cylindrical-shell method.

4. **(2 point bonus)** Calculate the volume of the solid produced by rotating the region shown in the above problem around the \( x \)-axis.

If we wish to use the disc/washer method, we would integrate with respect to \( x \) (a shell approach is possible as well). Here, the shape goes from \( x = 0 \) to \( x = 6 \), but changes shape radically at \( x = 4 \), so we would actually construct two separate integrals: when \( x \) is between 0 and 4, the outer radius is 3 and the inner radius is 2, while for \( x \) between 4 and 6, the outer radius is \( \frac{12}{x} \) and the inner radius is 2. Thus, to calculate the volume of the entire figure, we add the two integrals:

\[
\int_{0}^{4} \pi (3^2 - 2^2) dx + \int_{4}^{6} \pi \left( \frac{144}{x^2} - 2^2 \right) dx
\]