1. **(8 points)** Find the center of mass of the region bounded by the curves $y = x^3 + 2$ and $y = 4x + 2$. Expressions need not be arithmetically reduced.

To find the center of mass, we need first find the area $A$, $x$-moment $M_x$, and $y$-moment $M_y$:

$$A = \int_0^2 (4x + 2) - (x^3 + 2) \, dx = \int_0^2 4x - x^3 = 2x^2 - \frac{x^4}{4} \bigg|_0^2 = (8 - 4) - (0 - 0) = 4$$

$$M_x = \int_0^2 x[(4x + 2) - (x^3 + 2)] \, dx = \int_0^2 4x^2 - x^4 = \frac{4x^3}{3} - \frac{x^5}{5} \bigg|_0^2 = \frac{32}{3} - \frac{32}{5} = \frac{64}{15}$$

$$M_y = \int_0^2 \frac{1}{2}[(4x + 2)^2 - (x^3 + 2)^2] \, dx = \int_0^2 -\frac{1}{2}x^6 - 2x^3 + 8x^2 + 8x \, dx$$

$$= -\frac{x^7}{14} - \frac{x^4}{2} + \frac{8x^3}{3} + 4x^2 \bigg|_0^2 = -\frac{64}{7} - 8 + \frac{64}{3} + 16 = \frac{424}{21}$$

So the center of mass is $\left(\frac{M_x}{A}, \frac{M_y}{A}\right) = \left(\frac{16}{15}, \frac{108}{21}\right)$.

2. **(8 points)** Let $f(x) = \begin{cases} 0 & \text{for } x < 1 \\ \frac{2}{x^3} & \text{for } x \geq 1 \end{cases}$

(a) **(4 points)** Verify that $f(x)$ is a probability distribution function.

A cursory inspection reveals that this function is non-negative throughout: 0 is non-negative everywhere, and $\frac{2}{x^3}$ is non-negative as long as $x > 0$. The critical property to demonstrate that this function is a probability distribution function is simply that $\int_{-\infty}^\infty f(x) \, dx = 1$. We can simplify this somewhat by ignoring the region on which $f(x)$ is zero, so that $\int_{-\infty}^\infty f(x) = \int_1^\infty f(x) \, dx$. We evaluate this as such:
\[ \int_{1}^{\infty} f(x) \, dx = \int_{1}^{\infty} \frac{2}{x^{3}} \, dx \]
\[ = \lim_{b \to \infty} \int_{1}^{b} \frac{2}{x^{3}} \, dx \]
\[ = \lim_{b \to \infty} \int_{1}^{b} \frac{2}{x^{3}} \, dx \]
\[ = \lim_{b \to \infty} \left[ -\frac{1}{x^{2}} \right]_{1}^{b} \, dx \]
\[ = \lim_{b \to \infty} \left[ -\frac{1}{b^{2}} + \frac{1}{1^{2}} \right] \, dx \]
\[ = 1 \]

(b) (4 points) For a random variable \( X \) described by the above probability distribution function, find \( P(X \geq 10) \).

We proceed as above but calculating \( P(X \geq 10) \), which is \( \int_{10}^{\infty} f(x) \):

\[ \int_{10}^{\infty} f(x) \, dx = \int_{10}^{\infty} \frac{2}{x^{3}} \, dx \]
\[ = \lim_{b \to \infty} \int_{10}^{b} \frac{2}{x^{3}} \, dx \]
\[ = \lim_{b \to \infty} \left[ -\frac{1}{x^{2}} \right]_{10}^{b} \, dx \]
\[ = \lim_{b \to \infty} \left[ -\frac{1}{b^{2}} + \frac{1}{10^{2}} \right] \, dx \]
\[ = \frac{1}{100} \]

Thus, this random variable will only have a value greater than 10 in one out of every hundred tests.

3. (8 points) Answer the following questions about the differential equation \( \frac{dy}{dx} = y^{2} \).

(a) (4 points) Verify that \( y = \frac{1}{5-x} \) is a solution to this differential equation.

We evaluate each side, substituting \( \frac{1}{5-x} \) in for \( y \):

\[ \frac{dy}{dx} = \frac{d}{dx} \frac{1}{5-x} = \frac{d}{dx} (5-x)^{-1} = (-1) [-(5-x)^{-2}] = (5-x)^{-2} \]
\[ y^{2} = \left( \frac{1}{5-x} \right)^{2} = \frac{1}{(5-x)^{2}} = (5-x)^{-2} \]

Since these evaluations are identical, the assignment \( y = \frac{1}{5-x} \) indeed satisfies the differential equation.
(b) **(4 points)** Using Euler’s method, if \( y = 2 \) when \( x = 1 \), estimate the value of \( y \) when \( x = 1.2 \), using a step size of 0.1.

We have a differential equation whose slope (i.e. \( \frac{dy}{dx} \)) at each point is described by the function \( m(x, y) = y^2 \). We will be using Euler’s method on this with \( \Delta x = 0.1 \) and initial point of \((x_0, y_0) = (1, 2)\). From this, we will calculate new positions \( x_1 \) and \( y_1 \).

\[
x_1 = x_0 + \Delta x = 1 + 0.1 = 1.1
\]
\[
y_1 = y_0 + \Delta x m(x_0, y_0) = 2 + 0.1(2^2) = 2.4
\]

so the second point in our estimation of this curve is \((1.1, 2.4)\). We repeat Euler’s method at this new point to find \( x_2 \) and \( y_2 \):

\[
x_2 = x_1 + \Delta x = 1.1 + 0.2 = 1.2
\]
\[
y_2 = y_1 + \Delta x m(x_1, y_1) = 2.4 + 0.1(2.4^2) = 2.976
\]

so when \( x = 1.2 \), we estimate that \( y = 2.976 \).

Note: the actual solution to this differential equation with initial condition, as discussed in class, is \( y = \frac{1}{3 - x} \); when \( x = 1.2 \) the actual value of \( y \) is thus actually \( \frac{10}{3} \), which is not actually all that close to 2.976.